

Lecture 10 (Feb 22, 2016)

Lasalle Invariance Principle

Motivating Example: $\dot{x} = -y - x^3$ eq. pt $(0,0)$, $\frac{\partial f}{\partial x} \Big|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

$$\dot{y} = x^5$$

origin is non-hyperbolic eq. pt.

sketch of phase portrait shows orbits circling origin but not clear whether they spiral in or out on closed curves!

Let's try Lyapunov method:

1) $V = \frac{1}{2}(x^2 + y^2) \rightarrow \dot{V} = x\dot{x} + y\dot{y} = -xy - x^4 + 2x^5y$ many indefinite terms

2) consider a higher power in x to get rid of these:

$$V = x^6 + \alpha y^2 \rightarrow \dot{V} = (2\alpha - 6)x^5y - 6x^8 \xrightarrow{\alpha=3} \dot{V} = -6x^8 \leq 0$$

\dot{V} is only negative semi-definite, so can conclude Lyapunov stability.

Is the origin a.s.?

3) consider $V = x^6 + xy^3 + 3y^2$. Is V positive definite?

Young's Inequality

$$a, b \geq 0, \text{ then } ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

$\forall (p, q) \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Apply Y.I.: $a = |x|$, $b = |y|^3$, $p = 6$, $q = 5/6$ s.t. if $|y| < 1$,

$$|xy^3| = |x||y|^3 \leq \frac{|x|^6}{6} + \frac{5|y|^{18/5}}{6} \leq \frac{1}{6}x^6 + \frac{5}{6}y^2$$

$$\therefore V \geq \frac{5}{6}x^6 + \frac{13}{6}y^2$$

Similarly, one can show that if $|y| < 1/2$ then $\dot{V} \leq -\frac{27}{8}x^8 - \frac{21}{64}y^4$
 \Rightarrow The origin is a.s.

Would have been easier to stick with $V = x^6 + 3y^2$.

Note : $\{x : \dot{V}(x) = 0\} = \{x = 0\} = y\text{-axis}$

Further, on the y -axis the vector field is not in general parallel to the y -axis. (The solutions starting there, will leave)

LaSalle's Invariance Principle makes it possible to use such V to prove asymptotic stability!

Def Let $x(t)$ be a solution of $\dot{x} = f(x)$. A point P is "positive limit point" of $x(t)$ if $\exists \{t_n\}$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ s.t. $x(t_n) \xrightarrow[n \rightarrow \infty]{} P$.

Def. set of all positive limit points of $x(t)$ is positive limit set.

Def. Set M is "invariant" w.r.t. $\dot{x} = f(x)$ if $x(0) \in M \rightarrow x(t) \in M \quad \forall t \in \mathbb{R}$

Def. $x(t)$ approaches set M as t approaches infinity, if given any $\epsilon > 0$, $\exists T > 0$ s.t. $\text{dis}(x(T), M) < \epsilon$, $\forall t > T$, with $\text{dist}(P, M) = \inf_{x \in M} \|P - x\|$

Property of limit sets.

Lemma 4.1. If a solution $x(t)$ is bounded & $x(t) \in D$ for $t \geq 0$, then its positive limit set L^+ is nonempty, compact & invariant. Further, $x(t)$ approaches L^+ as $t \rightarrow \infty$.

LaSalle's Invariance Principle

Theorem 4.4. Let $\Omega \subset D$ be a compact set s.t. that is positively invariant with respect to $\dot{x} = f(x)$. Let $V: D \rightarrow \mathbb{R}$ be a C^1 function s.t. $\dot{V}(x) < 0$ in Ω . Let $E = \{x \in \Omega : \dot{V}(x) = 0\}$. Let M be the largest invariant subset in E . Then every solution starting in Ω approaches M as $t \rightarrow \infty$.

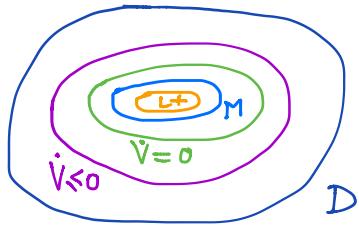
Proof. Let $x(t)$ be a solution starting in Ω .

Ω : positive invariant $\Rightarrow x(t) \in \Omega \quad \forall t \geq 0$

$\dot{V} \leq 0$ on $\Omega \Rightarrow V(x(t))$ is decreasing in t .

$V(x)$ is continuous on a compact set Ω

$\Rightarrow V$: bounded from below on $\Omega \Rightarrow \lim_{t \rightarrow \infty} V(x(t)) =: a$ exists.



The positive limit set L^+ of $x(t)$ is in Ω because Ω is a closed set (If Ω were open, L^+ could be on boundary of Ω).

Next we show that $\dot{V}=0$ in L^+ and conclude that $L^+ \subseteq E$.

For any $p \in L^+$, \exists a sequence $\{t_n\}$ with $t_n \rightarrow \infty$ & $x(t_n) \xrightarrow[n \rightarrow \infty]{} p$.

By continuity of $V(x)$, $a = \lim_{n \rightarrow \infty} V(x(t_n)) = V(\lim_{n \rightarrow \infty} x(t_n)) = V(p)$.

$\therefore V(x) \equiv a \quad \forall x \in L^+$.

Since L^+ is invariant (Lemma 4.1), L^+ is an invariant subset of E .

$\Rightarrow L^+ \subset M \subset E$.

Since $X(t)$ is bounded, $X(t) \rightarrow L^+$, as $t \rightarrow \infty$ (Lemma 4.1). Hence $X(t)$ approaches M as $t \rightarrow \infty$. ■

Note V does not have to be positive definite although it is useful.

E.g. if $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is bounded and $\dot{V}(x) \leq 0$ on Ω_c , then can choose $\Omega = \Omega_c$. Get boundedness for small enough c if V is positive definite.

Note To show asymptotic stability of origin, show M is the origin.

Note Can use Ω as estimate of region of attraction.

Theorems of Barbashin and Krasovskii (proved before LaSalle)

Generalization of Theorem 4.1 & 4.2

Corollary 4.1. Let $x=0$ be eq. pt. Let $V: D \rightarrow \mathbb{R}$ be a C^1 & positive definite function on D containing origin st. $\dot{V}(x) \leq 0$ in D . Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S other than $x(t)=0$. Then the origin is a.s.

In thm 4.1, $\dot{V} < 0 \rightarrow S = \{0\}$

Corollary 4.2. Let $x=0$ be eq. pt. & $V: D \rightarrow \mathbb{R}$ be a C^1 , radially unbounded & positive definite function on \mathbb{R}^n st. $\dot{V}(x) \leq 0$, $\forall x \in \mathbb{R}^n$. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S other than $x(t)=0$. Then the origin is g.a.s.

In thm 4.2. $\dot{V} < 0 \rightarrow S = \{0\}$

Revisit motivating example:

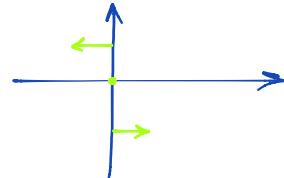
$$\dot{x} = -y - x^3$$

$$\dot{y} = x^5$$

$$V(x,y) = x^6 + 3y^2$$

V is positive definite and C^1 and radially unbounded

$$\dot{V} = -6x^8 \leq 0 \quad \forall (x,y) \in \mathbb{R}^2$$



$$S = \{(x,y) \in \mathbb{R}^2 \mid \dot{V}=0\} = \{(x,y) \mid -6x^8=0\} = \{(x,y) \in \mathbb{R}^2 \mid x=0\}$$

The dynamics on S :

$$\dot{V}=0 \Rightarrow x=0 \Rightarrow \dot{x}=0 \Rightarrow 0 = -y - 0^3 \Rightarrow y=0 \Rightarrow (x,y) = (0,0)$$

$\therefore (0,0)$: g.a.s.

Lyapunov design of adaptive controller

$$\dot{y} = ay + u \quad a: \text{unknown or slowly varying}$$

$$u = -ky, \quad k = \gamma y^2, \quad \gamma > 0 \quad \text{adaptive control law}$$

$$\text{Let } x_1 = y, \quad x_2 = k \quad \begin{cases} \dot{x}_1 = -(x_2 - a)x_1 \\ \dot{x}_2 = \gamma x_1^2 \end{cases}$$

The line $x_1=0$ is an eq. set. we want to show that trajectories approach this set as $t \rightarrow \infty$ which means the adaptive controller regulates y to zero.

We want to apply Thm 4.4.

□ consider $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2$ where $b > a$

$$\square \quad \dot{V} = x_1 \dot{x}_1 + \frac{1}{\gamma} (x_2 - b) \dot{x}_2 = -x_1^2(x_2 - a) + x_1^2(x_2 - b) = -x_1^2(b - a) \leq 0 \quad \forall x \in \mathbb{R}^2$$

□ $V(x)$ is radially unbounded & $\dot{V} \leq 0 \Rightarrow \Omega_c = \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$ is compact

& Positively invariant, for some arbitrary c . Fix c . Take $\Omega = \Omega_c$.

$$E = \{x \in \Omega \mid \dot{V} = 0\} = \{x \in \Omega \mid x_1 = 0\}$$

Because any point on the line $x_1=0$ is an eq. pt. E is invariant set

$$\Rightarrow M = E$$

Thm 4.4 Every trajectory starting in Ω approaches E as $t \rightarrow \infty$.

For any initial condition $x(0)$, choose c large enough st $x(0) \in \Omega_c$

(V : radially unbounded $\rightarrow c$ could be arbitrary large)

\Rightarrow The conclusion is global.